Global microlocalization of étale sheaves

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2026.11.19

In this lecture, for simplicity, we assume

- k is an algebraically closed field of char $p > 0$.
- Λ is a finite local ring with residue characteristic $\ell \neq p$.
- Fix a non-trivial character $\psi : \mathbb{F}_p \longrightarrow \Lambda^\times$. $\mathcal{L}_\psi$ is the Artin–Schreier sheaf on $\mathbb{A}^1$ associated to ψ.
- X is a smooth scheme over k of pure dimension d.
- η_0: generic point of henselization of $\mathbb{A}^1$ at 0.

In Saito–Xiong–Yang, we have constructed $\mu\mathcal{H}om(-,-)$ on T^*X. It is an object of

$$D_{\mathrm{ctf}}(T^*X \times_k \eta_0, \Lambda).$$

Motivated by the work of Kashiwara-Schapira-Ivorra-Waschkies: "Microlocalization of ind-sheaves", we can ask:

Question. Can we construct a covariant functor

$$\mu(-) : D_{\mathrm{ctf}}(X, \Lambda) \longrightarrow {}^{\text{Certain Categories}}_{\text{on } T^*X}$$

such that

$$R\mathcal{H}om\Big(\mu(\mathcal{F}), \mu(\mathcal{G})\Big) \simeq \mu\mathcal{H}om(\mathcal{F}, \mathcal{G}) \qquad \text{and} \qquad \mathrm{Supp}_k(\mu(\mathcal{F})) = SS(\mathcal{F})\ ?$$

For such a statement, we have to construct a target category, together with a composition law for $\mu\mathcal{H}om(-,-)$.

§ 1 Deformation of the pair groupoid $X^\bullet$

For a closed immersion $Z \hookrightarrow Y$ between smooth schemes, with ideal sheaf $\mathcal{I}$, we consider

$$\begin{array}{ccc} A_Z(X) = \mathrm{Spec}_{\mathcal{O}_Y} \mathcal{O}_Y[t, \mathcal{I}t^{-1}] & \xrightarrow[t]{\text{smooth}} & \mathbb{A}^1_t \\ \uparrow & \square & \uparrow \\ T_Z X & \longrightarrow & 0 \end{array}$$

We explain: The deformation interpolates between two groupoids

$$X \times X \rightrightarrows X, \qquad TX \rightrightarrows X \quad \text{(additive group structure)}.$$

For $n \geq 1$, let $\delta_n : X \longrightarrow X^{n+1}$ be the diagonal.

Put $A_0 = X \times \mathbb{A}^1$ and $A_n = A_X(X^{n+1})$ for $n \geq 1$.

The two projections

$$X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X$$

define two smooth morphisms

$$A_1 \underset{t}{\overset{s}{\rightrightarrows}} A_0,$$

called source map and target map.

s and t are maps over $\mathbb{A}^1$; over $0 \in \mathbb{A}^1$, they are

$$\begin{array}{ccc} TX & \rightrightarrows & X \times 0 \\ & \searrow \quad \swarrow & \\ & 0 & \end{array}$$

By [Saito–Xiong–Yang, Corollary 2.2.8], we have an isomorphism

$$\begin{array}{ccc} A_n & \xrightarrow{\sim} & A_1 \,{}_t\!\times_{A_0 s} \cdots {}_t\!\times_{A_0 s} A_1 \\ & {\scriptstyle q_n}\searrow \quad \swarrow{\scriptstyle \text{“two endpoints”}} & \\ & A_1 & \end{array} \qquad (*)$$

composition

$$\underbrace{1 \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet 2}_{\text{two endpoints}}$$

This is induced by

$$X^{n+1} \xrightarrow{\mathrm{pr}_{1,n+1}} X^2.$$

Over $0 \in \mathbb{A}^1$, the diagram $(*)$ is

$$\begin{array}{ccc} T_X(X^{n+1}) = (TX)^{n+1}/\operatorname{diag}(TX) & \xrightarrow{\sim} & \underbrace{TX \times_X \cdots \times_X TX}_{n\text{-factors}} \\ & \searrow \quad \swarrow & \\ & TX & \end{array}$$

$$(u_0, \ldots, u_n) \longmapsto (u_1 - u_0, \ldots, u_n - u_{n-1}), \qquad (v_1, \ldots, v_n) \longmapsto v_1 + v_2 + \cdots + v_n.$$

Generalization of $(*)$

Let $n_1, \ldots, n_r \geq 0$ and $N = \sum_j n_j$. The map

$$q_0 : A_0 = X \times \mathbb{A}^1 \longrightarrow A_1 = A_X(X \times X)$$

is regarded as the “identity section.” Let $r_0 : A_0 = X \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the projection. Then

$$A_N \simeq A_r \times_{A_1^r} \left(\prod_{j=1}^{r} A_{n_j} \right), \qquad A_r \xrightarrow{r_n} A_1^r,$$

where products are over $\mathbb{A}^1$.

Consider the correspondence

$$A_1^n \xleftarrow{r_n} A_n \xrightarrow{q_n} A_1. \tag{$**$}$$

Here A_1^n denotes the product over $\mathbb{A}^1$; r_n is the adjacent-pair map, and $q_n : A_n \to A_1$ is the endpoint map defined in $(*)$.

On the generic fiber, $(**)$ is

$$(X \times X)^n \longleftarrow X^{n+1} \xrightarrow{p_{0,n}} X \times X.$$

On the special fiber, $(**)$ is

$$\underset{\text{product over } k}{(TX)^n} \longleftarrow \underset{\text{product over } X}{(TX)^n_X} \xrightarrow{a_n} TX.$$

§ 2 Microlocal Hom

For the identity section $A_0 \to A_1$, its generic fiber is the diagonal, and its special fiber is $0 : X \to TX$. On $X \times X$, write

$$\mathcal{H}(F, G) := R\mathcal{H}om\Big(\mathrm{pr}_1^* F, \mathrm{pr}_2^! G\Big) \simeq D(F) \boxtimes G.$$

Specializing $\delta_! \Lambda \to \mathcal{H}(F, F)$ gives

$$0_! \Lambda_X \longrightarrow \nu\mathcal{H}om_X(F, F).$$

Microlocalization gives

$$u_F : \Lambda_{T^*X} \longrightarrow \mu\mathcal{H}om_X(F, F) \qquad [\text{unit}].$$

Proposition. Let $\pi : T^*X \to X$. There is a natural equivalence

$$R\pi_* \mu\mathcal{H}om_X(F, G) \simeq R\mathcal{H}om_X(F, G).$$

Consequently,

$$R\Gamma\Big(T^*X, \mu\mathcal{H}om_X(F, G)\Big) \simeq R\,\mathrm{Hom}_X(F, G).$$

Under the equivalence

$$R\Gamma\Big(T^*X, \mu\mathcal{H}om_X(F, F)\Big) \simeq R\,\mathrm{Hom}(F, F),$$

u_F maps to id_F.

Proof. Note that

$$R\pi_*\mathcal{F}_\psi \simeq 0^!_{TX}.$$

Indeed, for $\mathcal{L}$ on X and $\mathcal{K}$ on TX, we have

$$\begin{aligned} R\operatorname{Hom}_X\Big(\mathcal{L}, R\pi_*\mathcal{F}_\psi\mathcal{K}\Big) &\simeq R\operatorname{Hom}_{T^*X}\Big(\pi^*\mathcal{L}, \mathcal{F}_\psi\mathcal{K}\Big) \\ &\simeq R\operatorname{Hom}_{T^*X}\Big(\mathcal{F}_\psi 0_!\mathcal{L}, \mathcal{F}_\psi\mathcal{K}\Big) \\ &\overset{\mathcal{F}_\psi \text{ equiv.}}{\simeq} R\operatorname{Hom}_{TX}(0_!\mathcal{L}, \mathcal{K}) \\ &\simeq R\operatorname{Hom}_X(\mathcal{L}, 0^!\mathcal{K}). \end{aligned}$$

Then

$$\begin{aligned} R\pi_*\mathcal{F}_\psi\nu_{X/X\times X}\Big(\mathcal{H}(F,G)\Big) &\simeq 0^!_{TX}\nu_{X/X\times X}\Big(\mathcal{H}(F,G)\Big) \\ &\simeq \delta^!\mathcal{H}(F,G) \\ &\simeq R\mathcal{H}om_X(F,G). \end{aligned}$$

□

Consider

$$X^3 \underset{\longrightarrow}{\overset{\longrightarrow}{\overset{p_{01}}{\overset{p_{02}}{\overset{p_{12}}{\longrightarrow}}}}} X^2 \qquad \text{and} \qquad X^3 \underset{\longrightarrow}{\overset{\longrightarrow}{\overset{p_0}{\overset{p_1}{\overset{p_2}{\longrightarrow}}}}} X.$$

Then there is a cohomological correspondence

$$\begin{array}{ccc} p_{01}^*\mathcal{H}(F,G)\otimes p_{12}^*\mathcal{H}(G,H) & \longrightarrow & p_{02}^!\mathcal{H}(F,H) \\ \downarrow\simeq & & \downarrow\simeq \\ p_0^*D(F)\otimes p_1^*(G\otimes D(G))\otimes p_2^*H & \xrightarrow[\text{evaluation}]{} & p_0^*D(F)\otimes p_1^*K_X\otimes p_2^*H. \end{array}$$

Here the right-hand identification is

$$p_{02}^!\mathcal{H}(F,H) \simeq p_{02}^!\Big(D(F)\boxtimes H\Big) \simeq p_0^*D(F)\otimes p_1^*K_X\otimes p_2^*H.$$

Here we use the fact that p_{02} is smooth.

This gives a morphism

$$p_{02!}\Big(p_{01}^*\mathcal{H}(F,G)\otimes p_{12}^*\mathcal{H}(G,H)\Big) \longrightarrow \mathcal{H}(F,H).$$

For $F_0,\dots,F_n \in D_{\mathrm{ctf}}(X,\Lambda)$, the same construction gives

$$\bigotimes_{i=1}^{n} p_{i-1,i}^*\mathcal{H}(F_{i-1},F_i) \longrightarrow p_{0n}^!\mathcal{H}(F_0,F_n) \qquad \text{on } X^{n+1}. \tag{$***$}$$

Consider

$$A_1^n \xleftarrow{r_n} A_n \xrightarrow{q_n} A_1.$$

Applying specialization to $(***)$, we get

$$\begin{aligned} r_{n,s}^* \left(\boxtimes_{i=1}^n \nu\mathcal{H}om_X(F_{i-1}, F_i) \right) &\longrightarrow R\Psi_{A_n} \left(r_{n,\bar{\eta}}^* \boxtimes_{i=1}^n \mathcal{H}(F_{i-1}, F_i) \right) \\ &\xrightarrow{\text{ev}} R\Psi_{A_n} \left(q_{n,\bar{\eta}}^! \mathcal{H}(F_0, F_n) \right) \\ &\longrightarrow q_{n,s}^! \nu\mathcal{H}om_X(F_0, F_n). \end{aligned}$$

Adjunction gives (here $q_{n,s} = a_n$)

$$a_{n!} \left(\bigotimes_{i=1}^n \mathrm{pr}_i^* \, \nu\mathcal{H}om_X(F_{i-1}, F_i) \right) \longrightarrow \nu\mathcal{H}om_X(F_0, F_n).$$

Applying $\mathcal{F}_\psi$ gives

$$\bigotimes_{i=1}^n \mu\mathcal{H}om_X(F_{i-1}, F_i) \longrightarrow \mu\mathcal{H}om_X(F_0, F_n). \qquad (**)$$

Proposition. Put $\mathcal{V} = \mathrm{Ind}\big(D_{\mathrm{ctf}}(T^*X, \Lambda)\big)$. Consider $\mathcal{C} = D^b_{\mathrm{ctf}}(X, \Lambda)$ with Hom objects $\mu\mathcal{H}om_X(F, G)$:

$$\mathrm{Hom}_{\mathcal{C}}(F, G) := \mu\mathcal{H}om_X(F, G).$$

The operations $(**)$ define a $\mathcal{V}$-enriched category $\mathscr{A}_X$.

§ 3 Enriched Yoneda lemma and localization

A right $\mathscr{A}_X$-module M is a functor

$$\begin{aligned} M : \mathscr{A}_X^{\mathrm{op}} &\longrightarrow \mathcal{V}, \\ A &\longmapsto M(A), \end{aligned}$$

together with action maps

$$\mathscr{A}_X(A, B) \otimes M(B) \longrightarrow M(A).$$

The actions are associative and unital with all higher coherences.

For $F \in \mathscr{A}_X$, define

$$\begin{aligned} \mu(F) = \mathscr{A}_X(-, F) : \mathscr{A}_X^{\mathrm{op}} &\longrightarrow \mathcal{V}, \\ G &\longmapsto \mu(F)(G) = \mu\mathcal{H}om_X(G, F). \end{aligned}$$

Its action is the composition

$$\mathscr{A}(A, B) \otimes \mathscr{A}(B, F) \longrightarrow \mathscr{A}(A, F).$$

This defines a coherent functor

$$\mu : \mathcal{C} \longrightarrow \mathrm{Mod}_{\mathscr{A}_X}(\mathcal{C}),$$

which is exact and fully faithful.

Enriched Yoneda lemma. For every module N and every $F \in \mathscr{A}_X$, evaluation at the identity of F gives a natural equivalence

$$\underline{R\operatorname{Hom}}_{\mathscr{A}_X}\Big(\mu(F), N\Big) \simeq N(F).$$

Taking $N = \mu(G)$, we get

$$\underline{R\operatorname{Hom}}_{\mathscr{A}_X}\Big(\mu(F), \mu(G)\Big) \simeq \mu(G)(F) = \mu\mathcal{H}om_X(F, G).$$

Proposition. For every open $U \subseteq T^*X$,

$$\mu(F)|_U = 0 \quad \Longleftrightarrow \quad \mu\mathcal{H}om_X(F, F)|_U = 0.$$

In particular, the support of the module $\mu(F)$, defined as the complement of its largest open vanishing locus, is $\mathrm{SS}(F)$:

$$\operatorname{Supp}\mu(F) = \mathrm{SS}(F) = \operatorname{Supp}_k \mu\mathcal{H}om(F, F).$$

Corollary. If $F, G \in \mathcal{C}$, we have

$$\operatorname{Supp}_k \mu\mathcal{H}om_X(F, G) \subseteq \mathrm{SS}(F) \cap \mathrm{SS}(G).$$

Localization. For an open $U \subseteq T^*X$, put

$$\mathcal{C}_0(U) = \{F \in \mathcal{C} \mid U \cap \mathrm{SS}(F) = \varnothing\}.$$

This is a thick stable subcategory. The exact functor $F \mapsto \mu(F)|_U$ factors through the stable Verdier quotient

$$\mathcal{C}/\mathcal{C}_0(U) \longrightarrow \operatorname{Mod}_{\mathscr{A}_X|_U}\Big(\operatorname{Ind}(D^b_{\mathrm{ctf}}(U, \Lambda))\Big).$$

Here is the main reference:

[Saito-Xiong-Yang] Takeshi Saito, Jiangnan Xiong and Enlin Yang, *Singular supports, characteristic cycles, and microlocalization of constructible sheaves*, arXiv:2609.06625.